1969

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THESIS

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October 1969

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Design of Constrained Optimal Controls for Linear Regulators with Incomplete State Feedback

by

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Submitted in partial fulfillment of the
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ELECTRICAL ENGINEER

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October 1969

ABSTRACT

The need for an auxiliary performance measure for the design of constrained optimal controls for linear regulators is shown. Several auxiliary performance measures are compared, and the maximum of the absolute degradation over the admissible initial states is selected as the auxiliary performance measure. Computational algorithms which make extensive use of existing library subprograms are developed for the design of constrained optimal controls in those cases where the control vector is specified as constant or piecewise-constant linear feedback of the output vector. Numerical examples including a third-order system and a time-varying system are given to illustrate the applications of the proposed algorithms.

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ACKNOWLEDGEMENT

The author wishes to express his sincere appreciation to Professor Donald E. Kirk for his guidance and advice in the preparation of this thesis, and to Professor James S. Demetry for his helpful suggestions and encouragement.

I. INTRODUCTION

The linear regulator is one of the most extensively studied and well-known problems of optimal control theory [1,2,3,4]. The importance of linear regulators comes from the fact that many practical control problems can be formulated in the form of a linear regulator. In addition, the feedback form of the optimal solution to the problem is a very desirable feature. The practical application of the optimal solution to the linear regulator problem suffers from two serious drawbacks. One is the time-varying nature of the feedback gains, and the other is the need for exact physical measurements of the value of the state vector at every instant of time. In general, not the state vector of the system, but the output vector of the system is available for measurements. It is possible to reconstruct the state vector [5], or to obtain an approximation to the state vector that is best in some sense when noise is present [6] ; however, in many cases, these schemes to obtain the state vector are not justified due to economic considerations.

Since the mathematical formulation and solution of the optimal linear regulator problem by Kalman [2], the design of suboptimal controls, which are easy to implement, yet not too inferior to optimal ones has attracted the attention of several investigators.

Schoenberger [7] has given solutions to a class of problems in which the initial conditions are known. He has

also presented a procedure for minimizing the expected value of the performance measure when the probability distribution of the initial conditions is known.

Meditch [8] has proposed an approximate method of decoupling complex systems which simplifies the computation of the time-varying feedback gains.

Koivuniemi [9] has specified the form of the feedback control and has determined the unknown parameters to minimize the maximum degradation of the system performance over the admissible initial states.

Rekasius [10] has proposed a control law that minimizes the maximum (with respect to all initial states) relative deviation in the value of the performance measure with respect to the optimal performance measure. Although his method avoids some of the difficulties related to minimax problems, it is impractical for systems of higher than the second order.

In Kleinman and Athans' [11] formulation of the suboptimal linear regulator problem, easily realizable time functions are used to generate the suboptimal time-varying feedback gains. The unknown parameters are determined to minimize the expected value of the performance measure over the admissible initial states, which are assumed to be uniformly distributed on a unit hypersphere.

Kleinman, Fortmann and Athans [12] have given an application of Kleinman and Athans' [11] approach. They

have presented an algorithm for the numerical solution of the problem, for the case where the system is time-invariant and the suboptimal feedback gains are constrained to be piecewise constant.

Levine and Athans [13] have also used an average performance measure, but they have let the control vector to be a time-varying function of the system output rather than the system state.

Salmon [14] has proposed an algorithm which can be applied to the design of controllers for systems with unknown parameters, including initial conditions. In this method, the controller parameters are found in such a way that the maximum of either the system performance measure or the so-called performance sensitivity with respect to the unknown system parameters is minimized. The performance sensitivity is defined as the percentage or the absolute increase in the performance measure from that of an ideal controller caused by the unknown system parameters. The ideal controller is defined as the controller which is capable of measuring unknown system parameters and generating its own parameters accordingly.

In this thesis, after a brief review of the theory of optimal control of linear regulators, the need for an auxiliary performance measure for the formulation of the constrained optimal control problem is shown and several auxiliary performance measures are compared using a simple second-order example. Then the constrained optimal control

problem is formulated and a numerical method of solution for the case of constant feedback gains is given. Next, an approximate, but very practical, solution to the problem for the case of piecewise-constant feedback gains is proposed. Numerical examples that illustrate the properties of the techniques are given.

II. OPTIMAL CONTROL OF LINEAR REGULATORS

Consider the linear, time-varying system

$$\dot{\underline{x}}(t) = \underline{A}(t)\underline{x}(t) + \underline{B}(t)\underline{u}(t) \tag{1}$$

$$\underline{y}(t) = \underline{C}(t)\underline{x}(t) \tag{2}$$

where $\underline{x}(t)$ is the n-dimensional state vector, $\underline{u}(t)$ is the m-dimensional unconstrained control vector, $\underline{y}(t)$ is the r-dimensional output vector, and $\underline{A}(t)$, $\underline{B}(t)$ and $\underline{C}(t)$ are time-varying matrices of dimensions n x n, n x m, and r x n, respectively. Determine the control $\underline{u}^*(\cdot)$ that minimizes the performance measure

$$J(\underline{x}_o, t_o, \underline{u}(\cdot)) = \frac{1}{2}\,\underline{x}^T(t_f)\underline{H}\underline{x}(t_f)$$

$$+ \frac{1}{2}\int_{t_o}^{t_f} [\underline{x}^T(t)\underline{Q}(t)\underline{x}(t) + \underline{u}^T(t)\underline{R}(t)\underline{u}(t)]dt \tag{3}$$

where t_o and $t_f > t_o$ are the initial and final times, respectively, $\underline{x}_o = \underline{x}(t_o)$ is the initial state, $\underline{H}$ is a real symmetric positive semi-definite matrix, $\underline{Q}(t)$ is a time-varying real symmetric positive semi-definite matrix, $\underline{R}(t)$ is a time-varying real symmetric positive definite matrix, and the final state $\underline{x}(t_f)$ is free.

The solution to this problem is well-known [1,2,3,4]. The optimal control exists and is given by

$$\underline{u}^*(t) = -\underline{R}^{-1}(t)B^T(t)\underline{K}(t)\underline{x}(t)$$

$$\triangleq -\underline{F}^*(t)\underline{x}(t) \tag{4}$$

where $\underline{K}(t)$ is the unique symmetric positive definite solution of a matrix differential equation of the Riccati type given by

$$\dot{\underline{K}}(t) = -\underline{K}(t)\underline{A}(t) - \underline{A}^T(t)\underline{K}(t) - \underline{Q}(t)$$

$$+ \underline{K}(t)\underline{B}(t)\underline{R}^{-1}(t)\underline{B}^T(t)\underline{K}(t) \tag{5}$$

with the boundary condition

$$\underline{K}(t_f) = \underline{H}\ . \tag{6}$$

The state of the optimal system is the solution of

$$\dot{\underline{x}}(t) = [\underline{A}(t) - \underline{B}(t)\underline{F}^*(t)]\underline{x}(t) \tag{7}$$

with the boundary condition $\underline{x}(t_o) = \underline{x}_o$. The optimal performance measure is given by

$$J^*(\underline{x}_o,t_o) \triangleq J(\underline{x}_o,t_o,\underline{u}(\cdot))\Big|_{\underline{u}(\cdot)\,=\,\underline{u}^*(\cdot)}$$

$$= \frac{1}{2}\,\underline{x}_o^T\,\underline{K}(t_o)\underline{x}_o. \tag{8}$$

The matrix $\underline{K}(t)$ has another interesting property. For arbitrary $t \le t_f$ and $\underline{x}(t)$

$$J^*(\underline{x}(t),t) = \frac{1}{2}\,\underline{x}^T(t)\underline{K}(t)\underline{x}(t) \tag{9}$$

Some comments on the results just given are in order: first, the optimal control is a linear, time-varying feedback of the system states. (See Fig. 1). Second, the matrix Riccati differential equation, Eq. (5), is a set of n^2 coupled, first-order, nonlinear differential equations which can be solved numerically beginning at $t = t_f$ and integrating backward in time to $t = t_o$ using the boundary condition (6). Actually, only $n(n+1)/2$ of these equations must be solved because $\underline{K}(t)$ is symmetric.

In implementing the controller block in Fig. 1, the most appealing scheme is to calculate $\underline{K}(t_o)$ by integrating (5) backward off-line and then to calculate $\underline{F}^*(t)$ on-line by integrating (5) in the forward direction using previously calculated $\underline{K}(t_o)$. However, this is not possible in the time-varying case because of the instability of (5) in the forward time direction [2]. Even if the system is time-invariant, on-line integration is not always desirable due to increased complexity of the system. Another possibility is to provide storage for the elements of the feedback gain matrix $\underline{F}^*(t)$ and to store the precalculated values. This is possible because the elements of $\underline{F}^*(t)$ do not depend on the system state but depend only on time; therefore,

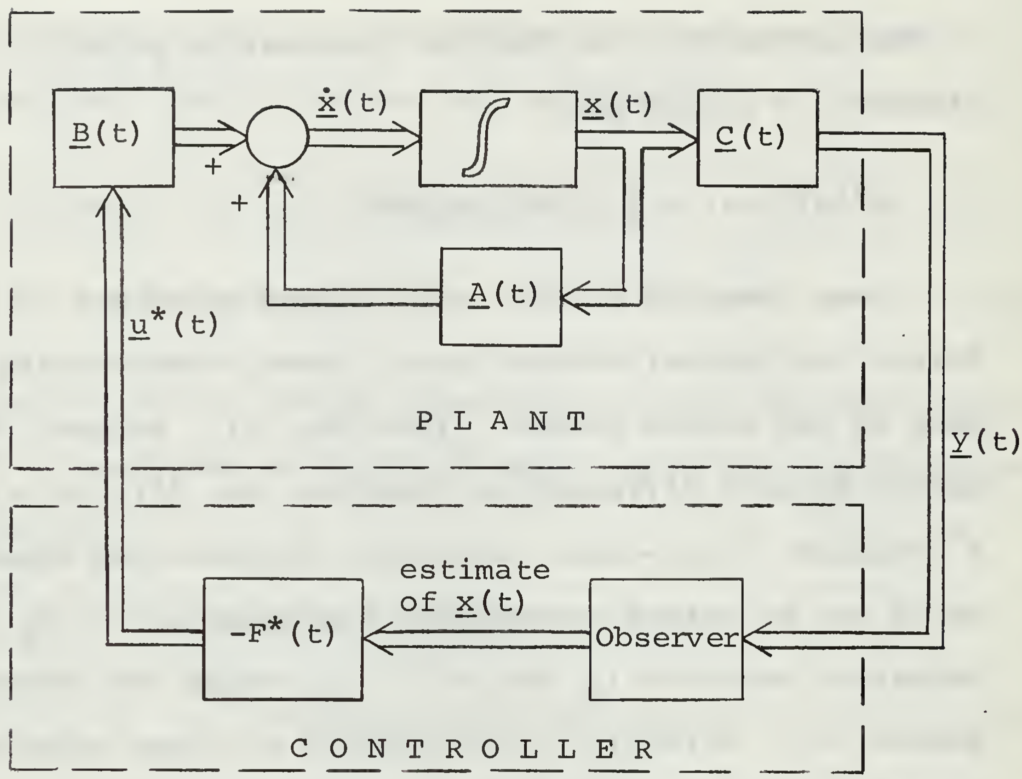

Fig. 1. Linear regulator with optimal control.

a priori knowledge of the initial state is not necessary. However, this scheme suffers from the same drawback as the previous scheme, namely, increased complexity.

Suppose that $\underline{F}^*(t)$ is readily available. Even then the realization of the optimal control, $\underline{u}^*(t)$, requires complete knowledge of the system state $\underline{x}(t)$. Reconstructing or estimating the state vector [5,6] from the knowledge of the output, $\underline{y}(t)$, may add additional complexity to the controller.

In many instances it is possible to decrease the complexity of a control system considerably by constraining the controller configuration and allowing a small increase

in the system performance measure with respect to optimal. If the adjustable parameters of such a controller are determined in some optimal fashion, then the control signal is called a constrained optimal control. In the following sections the design of constrained optimal controls for systems which can be modeled in the form of linear regulators will be discussed.

III. FORMULATION OF THE CONSTRAINED OPTIMAL CONTROL PROBLEM

For the system and the performance measure given by (1), (2) and (3) consider the control law

$$\underline{u}(t) = -\underline{P}(t)\underline{y}(t) \tag{10}$$

where $\underline{P}(t)$ is an m x r matrix which will be called the output-feedback gain matrix. The physical meaning of this control is as follows: the control signal is generated from the system output vector rather than from the system state vector as in the optimal control. Substituting (2) in (10), one obtains

$$\begin{aligned} \underline{u}(t) &= -\underline{P}(t)\underline{C}(t)\underline{x}(t) \\ &\triangleq -\underline{F}(t)\underline{x}(t) \end{aligned} \tag{11}$$

The value of the performance measure for the control law (11) is defined as

$$J(\underline{x}_o, t_o, \underline{P}(\cdot)) \triangleq J(\underline{x}_o, t_o, \underline{u}(\cdot))\Big|_{\underline{u}(\cdot) = -\underline{P}(\cdot)\underline{y}(\cdot)} \tag{12}$$

If $\underline{F}(t)$ defined by (11) is not equal to $\underline{F}^*(t)$ defined by (4) for all $t_o \le t \le t_f$, then the control (10) is not optimal. In general,

$$J(\underline{x}_o, t_o, \underline{P}(\cdot)) \ge J^*(\underline{x}_o, t_o) \tag{13}$$

It is shown in the Appendix that

$$J(\underline{x}_o, t_o, \underline{P}(\cdot)) = \frac{1}{2}\, \underline{x}_o^T\, \underline{V}(t_o, \underline{P}(\cdot))\underline{x}_o \;, \tag{14}$$

where $\underline{V}(t)$[1] is the symmetric, positive definite solution of the matrix differential equation

$$\begin{aligned}\dot{\underline{V}}(t) = - \underline{V}(t)[\underline{A}(t) &- \underline{B}(t)\underline{P}(t)\underline{C}(t)] \\ -[\underline{A}(t) &- \underline{B}(t)\underline{P}(t)\underline{C}(t)]^T\, \underline{V}(t) \\ - \underline{Q}(t) &- \underline{C}^T(t)\underline{P}^T(t)\underline{R}(t)\underline{P}(t)\underline{C}(t)\end{aligned} \tag{15}$$

with the boundary condition

$$\underline{V}(t_f) = \underline{H} \tag{16}$$

From the point of view of implementation, the control (10) has distinct advantages over the optimal control: first, knowledge of the system state vector $\underline{x}(t)$ is not needed. Second, constraints may be imposed on $\underline{P}(t)$ to make the structure of the control simple, e.g., $\underline{P}(t)$ may be required to be constant or piecewise constant.

A. NEED FOR AN AUXILIARY PERFORMANCE MEASURE

It is reasonable to suggest that the elements of $\underline{P}(t)$ be chosen so that the performance measure (12) is minimized,

---

[1] Note that $\underline{V}(t)$ depends on the values that $\underline{P}(t)$ attains in the interval between t and t_f . However this will be shown explicitly only when $t = t_o$ as in Eq. (14).

and at the same time the constraints imposed on the elements of $\underline{P}(t)$ are satisfied. However, there is one major difficulty in proceeding in this fashion: it is the need for a priori knowledge of the initial state of the system. It was pointed out in the previous section that although $J^*(\underline{x}_o,t_o)$ is a function of the initial state, the optimal feedback gain matrix $\underline{F}^*(t)$ is independent of the initial state. However, this is not the case with the control (10), in general. Not only $J(\underline{x}_o,t_o,\underline{P}(\cdot))$, but also the best output-feedback gain matrix, $\underline{P}(t)$, depends on the initial state. If the initial state is known a priori, then it is possible to find optimal values for the elements of $\underline{P}(t)$, for $t_o \le t \le t_f$, satisfying constraints of whatever complexity that may have been imposed on them, by minimizing the performance measure at least numerically, if not analytically. If the initial state is not known, then one approach is to define an auxiliary performance measure which is independent of the initial state. Some concepts that will be needed in defining an appropriate auxiliary performance measure will be given first.

The <u>Absolute Degradation</u>, denoted by AD, will be defined as

$$AD(\underline{x}_o,t_o,\underline{P}(\cdot)) \triangleq J(\underline{x}_o,t_o,\underline{P}(\cdot)) - J^*(\underline{x}_o,t_o) , \qquad (17)$$

that is, the absolute degradation is the increase in the performance measure with respect to the optimal value caused by application of a control given by (10).

The Relative Degradation, denoted by RD, is given by

$$RD(\underline{x}_o, t_o, \underline{P}(\cdot)) \triangleq \frac{J(\underline{x}_o, t_o, \underline{P}(\cdot)) - J^*(\underline{x}_o, t_o)}{J^*(\underline{x}_o, t_o)} \tag{18}$$

Both the absolute and relative degradations are always non-negative quantities.

Assuming that the possible initial states may lie within a hyperspherical region, with center at the origin and having a specified radius, there is no loss of generality in further assuming that the set of admissible initial states, denoted by X, consists of all points which lie on the unit hypersphere. This is because, for any admissible $\underline{x}_o$ and any $\rho \geq 0$,

$$J(\rho\underline{x}_o, t_o, \underline{u}(\cdot)) = \frac{1}{2} \rho^2 \underline{x}_o^T \underline{K}(t_o)\underline{x}_o$$

$$= \rho^2 J(\underline{x}_o, t_o, \underline{u}(\cdot)) , \tag{19}$$

i.e., the performance measure due to an initial state which is not on the unit hypersphere is a constant times the performance measure due to the corresponding point on the unit hypersphere.

There are two basic approaches in choosing an auxiliary performance measure: one is the statistical approach, the other is the worst initial state approach. In the first approach, it is assumed that the probability distribution of the initial state is known, and a statistical function of the performance measure that is independent of the initial

state is taken as the auxiliary performance measure. Several authors [7,11,12,13] use the expected value of the performance measure over the admissible initial states as the auxiliary performance measure. Normally, a simple probability density function, such as a uniform distribution is assumed. This makes the expected value of the performance measure a very simple function of the $\underline{V}$ matrix found from (15). Several variations on this approach depending upon the constraints imposed on the structure of the control and the assumed admissible initial states are given in the references cited above.

In the worst initial state approach [9,10,14], the maximum (worst) over the admissible initial states of any one of three basic quantities - performance measure, absolute degradation, and relative degradation - can be taken as the auxiliary performance measure. Therefore, there are four functionals of $\underline{P}(\cdot)$ which can be used as the auxiliary performance measure. These are:

$$J_1(\underline{P}(\cdot)) = \underset{\underline{x}_o \in X}{E} \{J(\underline{x}_o, t_o, \underline{P}(\cdot))\} \tag{20}$$

$$J_2(\underline{P}(\cdot)) = \underset{\underline{x}_o \in X}{\text{Max}}\ J(\underline{x}_o, t_o, \underline{P}(\cdot)) \tag{21}$$

$$J_3(\underline{P}(\cdot)) = \underset{\underline{x}_o \in X}{\text{Max}}\ AD(\underline{x}_o, t_o, \underline{P}(\cdot)) \tag{22}$$

$$J_4(\underline{P}(\cdot)) = \underset{\underline{x}_o \in X}{\text{Max}}\ RD(\underline{x}_o, t_o, \underline{P}(\cdot)) \tag{23}$$

The output-feedback gain matrix, which minimizes the auxiliary performance measure is called the constrained optimal output-feedback gain matrix and is denoted by $\underline{P}_s(t)$. The constrained optimal control, denoted by $\underline{u}_s(t)$, is given by (10) with $\underline{P}(t)$ replaced by $\underline{P}_s(t)$.

The performance measure resulting from the application of a constrained optimal control is called the constrained optimal performance measure and is denoted by $J_s(\underline{x}_o, t_o)$.

B. A COMPARISON OF SEVERAL AUXILIARY PERFORMANCE MEASURES

1. Example 1

The purpose of this example is to illustrate the differences that result from the use of various auxiliary performance measures. The method of obtaining the solutions will not be discussed, but rather the implications of each different result will be indicated.

The system to be controlled is described by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad . \tag{24}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad . \tag{25}$$

The performance measure for optimal control is

$$J(\underline{x}_o, u(\cdot)) = \int_o^2 [x_1^2(t) + u^2(t)]dt \quad . \tag{26}$$

The constrained optimal control is to be in the form

$$u_s(t) = -\begin{bmatrix} p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$$= -\begin{bmatrix} p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad (27)$$

where p_{11} and p_{12} are constants that minimize the auxiliary performance measure.

This problem was solved four times, using the four different auxiliary performance measures given by (20) - (23). The values obtained for p_{11} and p_{12} for each case are tabulated in Table 1. The conclusion that can be drawn looking at that table is very important: The values obtained for the parameters of the constrained optimal control depend very much on the auxiliary performance measure used. Therefore, the proper choice of an auxiliary performance measure is very significant. The values that the several performance quantities attain as functions of the initial state are also different in each case. Since the system is only second order and the initial state is assumed to lie on the unit circle, just one quantity, namely the ratio of the two components of the initial state vector, $x_2(t_o)/x_1(t_o)$,

Auxiliary Performance Measure	P_{11}	P_{12}
$\underset{\underline{x}_o \in X}{\mathrm{Max}}\ RD(\underline{x}_o, t_o, \underline{P})$	0.3055	0.5932
$\underset{\underline{x}_o \in X}{E}\ \{J(\underline{x}_o, t_o, \underline{P})\}$	0.8096	1.1668
$\underset{\underline{x}_o \in X}{\mathrm{Max}}\ J(\underline{x}_o, t_o, \underline{P})$	0.7312	1.5375
$\underset{\underline{x}_o \in X}{\mathrm{Max}}\ AD(\underline{x}_o, t_o, \underline{P})$	0.5643	0.9463

Table 1. Constrained optimal feedback gains for different auxiliary performance measures used in Ex.1.

is enough to specify a particular initial state[1]. The optimal performance measure J^*, constrained optimal performance measure J_s, absolute degradation AD, and relative degradation RD are plotted as functions of $x_2(t_o)/x_1(t_o)$ on one graph for each of the auxiliary performance measures in Figs. 2, 3, 4 and 5.

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1 It does not make any difference whether $x_2(t_o)$ and $x_1(t_o)$ are both in the first quadrant or both in the third quadrant (similarly for the second and fourth quadrants) due to symmetry.

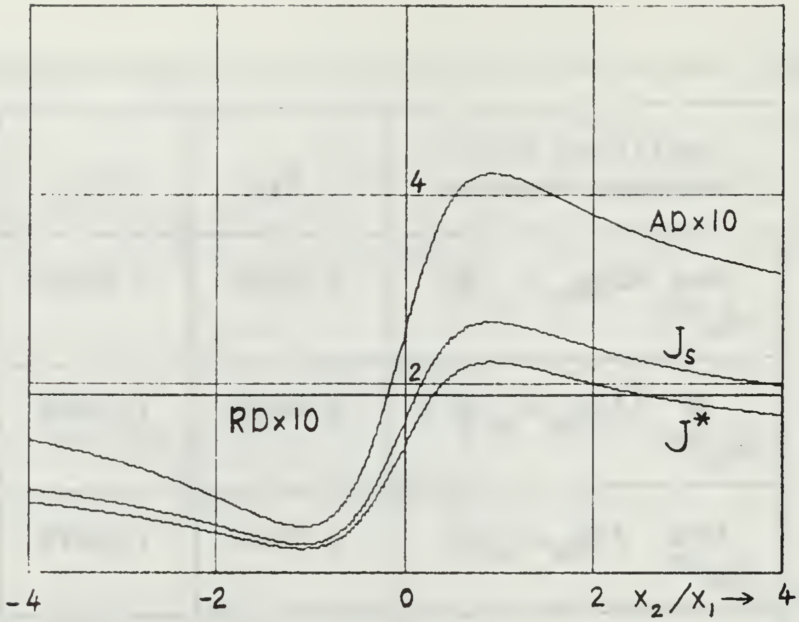

Fig. 2. Various performance values as functions of initial state. (Maximum relative degradation minimized.)

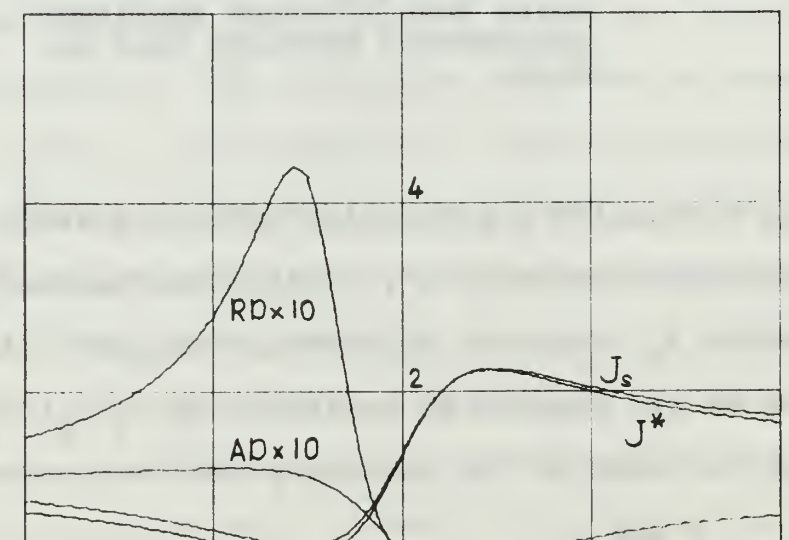

Fig. 3. Various performance values as functions of initial state. (Expected value of performance measure minimized.)

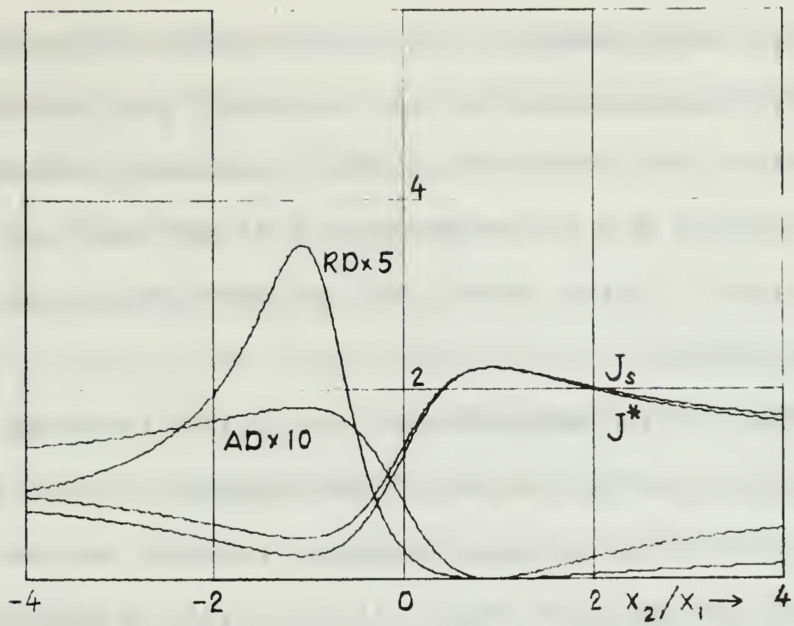

Fig. 4. Various performance values as functions of initial state. (Maximum performance measure minimized.)

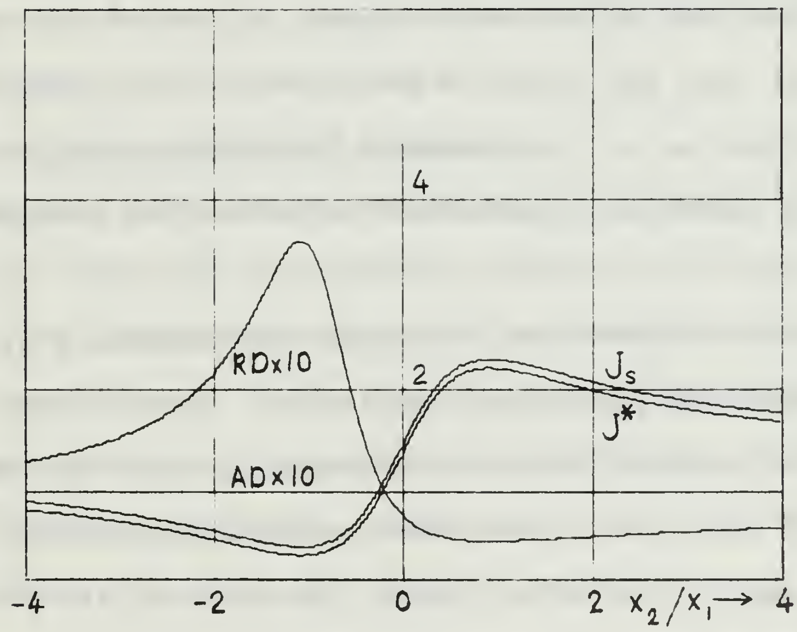

Fig. 5. Various performance values as functions of initial state. (Maximum absolute degradation minimized.)

For this example, it is clear that using the maximum relative degradation as the auxiliary performance measure (Fig. 1) is not very desirable. Although the maximum RD is less than 0.2 it amounts to a large absolute degradation for initial states where the optimal performance measure is already high.

When it is assumed that the initial states are uniformly distributed on the unit hypersphere and the expected value of the performance measure is used as the auxiliary performance measure (Fig. 3) the relative degradation goes as high as 0.43. But this occurs for initial states where the optimal performance measure is quite low, and the absolute degradation does not amount to a large value. In this case the J_s curve is closer to the J^* curve on the average than the other three cases. But, unless separately calculated as in this example, nothing can be said about maximum absolute degradation which may be more than allowable.

Fig. 4 shows the case when the maximum J is used as the auxiliary performance measure. In this case the constrained optimal control is almost as good as the optimal control when the constrained optimal performance measure is the maximum. In other words, the control is the best (closest to the optimal) when the initial condition is the "worst". This is a very desirable feature. However, it results in a very pessimistic design, i.e., the absolute

degradation for the other initial states becomes quite high. As far as the absolute degradation is concerned, allowing a small increase for the worst initial state may result in a significant improvement for the other initial states, as is the case in Fig. 5.

In the last case where the maximum absolute degradation is used as the auxiliary performance measure (Fig. 5), the constrained optimal performance measure as a function of the initial state is made as close to the optimal performance measure as possible by minimizing the maximum difference between the two, i.e., the maximum absolute degradation. When this approach is used one is assured that the system performance will not be degraded more than a certain amount with respect to the optimal system performance no matter what the initial state is.

2. Some Comments on Example 1

Example 1 has demonstrated that the choice of an appropriate auxiliary performance measure is a significant step in the design of constrained optimal controls for linear regulators. It is easy to reject the maximum relative degradation as the auxiliary performance measure in almost all applications. After all, one would not want to get farther away from the optimal performance as the control task becomes more demanding. The remaining three definitions for the auxiliary performance measure are all reasonable and each one of them can be used depending on the application. During the remainder of this work, the

maximum absolute degradation will be used as the auxiliary performance measure. The results obtained can be easily applied to the case where the maximum performance measure is used as the auxiliary performance measure. The case where the expected value of the performance measure is used as the auxiliary performance measure is easier to handle mathematically (especially when the probability distribution of the initial state is assumed to be uniform on the unit hypersphere) and is treated elsewhere [7,11,12] .

C. STATEMENT OF THE CONSTRAINED OPTIMAL CONTROL PROBLEM

It is assumed that the system is described by the state and output equations

$$\underline{\dot{x}}(t) = \underline{A}(t)\underline{x}(t) + \underline{B}(t)\underline{u}(t) \tag{1}$$

$$\underline{y}(t) = \underline{C}(t)\underline{x}(t) \tag{2}$$

given previously. The performance measure for optimal control is

$$J(\underline{x}_o, t_o, \underline{u}(\cdot)) = \frac{1}{2}\, \underline{x}^T(t_f)\underline{H}\, \underline{x}(t_f) + \frac{1}{2}\int_{t_o}^{t_f} [\underline{x}^T(t)\underline{Q}(t)\underline{x}(t) + \underline{u}^T(t)\underline{R}(t)\underline{u}(t)]dt \, . \tag{3}$$

Let the set of admissible initial states be X, and let X consist of the points which lie on the unit hypersphere. Let Ω denote a specified class of real time functions.

The constrained optimal control problem is to find a control

$$\underline{u}_s(t) = -\underline{P}_s(t)\underline{y}(t) \tag{28}$$

such that the elements of $\underline{P}_s(t)$ are members of Ω and the auxiliary performance measure

$$J_a(\underline{P}(\cdot)) = \underset{\underline{x}_o \in X}{\text{Max}}\ AD(\underline{x}_o, t_o, \underline{P}(\cdot)) \tag{29}$$

is minimized.

The value of the auxiliary performance measure corresponding to $\underline{P}_s(t)$ is denoted by J_a^*, i.e.,

$$J_a^* = J_a(\underline{P}(\cdot))\Big|_{\underline{P}(\cdot) = \underline{P}_s(\cdot)}$$

$$= \underset{\underline{P}(\cdot)\in\Omega}{\text{Min}}\ J_a(\underline{P}(\cdot)) \ . \tag{30}$$

The magnitude of J_a^* is an indication of how close the constrained optimal control is to the optimal control. However, a normalized number is usually more meaningful in making comparisons; therefore, the Normalized Degradation of the system, denoted by ND, will be defined as

$$ND = \frac{J_a^*}{\underset{\underline{x}_o \in X}{\text{Max}}\ J^*(\underline{x}_o, t_o)} \ . \tag{31}$$

ND is zero if the performance of the constrained optimal control is the same as the performance of the optimal control for all initial states; otherwise, it will be a positive number.

IV. SOLUTION OF THE CONSTRAINED OPTIMAL CONTROL PROBLEM

A. CONSTANT OUTPUT-FEEDBACK GAIN MATRIX

In this section, the constrained optimal control problem will be solved for the case where the elements of $\underline{P}(t)$ are constrained to be constant in the interval between t_o and t_f .

1. Development of the Method

It is desired that the constant matrix $\underline{P}_s$ be found such that

$$J_a(\underline{P}_s) = J_a^*$$

$$= \underset{\underline{P}}{\text{Min}} \ \underset{\underline{x}_o \epsilon X}{\text{Max}} \ AD(\underline{x}_o, t_o, \underline{P}) \ . \tag{32}$$

This is a minimax problem, and there are many difficulties associated with the solution of such problems: first, in general, an iterative numerical procedure in which a complete maximization process is required for each minimization step is necessary [9]; second, no matter how smooth the function $AD(\underline{x}_o, \underline{P})$ is in both $\underline{x}_o$ and $\underline{P}$, the function

$$J_a(\underline{P}) = \underset{\underline{x}_o \epsilon X}{\text{Max}} \ AD(\underline{x}_o, t_o, \underline{P}) \tag{33}$$

is not, in general, differentiable on the elements of $\underline{P}$. In particular, it may not be differentiable in $\underline{P}$ at the point yielding the minimum [14,15]. Therefore, some

method which does not require the partial derivatives of $J_a(\underline{P})$ with respect to the elements of $\underline{P}$ is more suitable than, for example, a method in which the gradient vector is needed and the partial derivatives are evaluated by making small perturbations.

It is possible to simplify the procedure considerably by noting that

$$\begin{aligned} AD(\underline{x}_o, t_o, \underline{P}) &= J(\underline{x}_o, t_o, \underline{P}) - J^*(\underline{x}_o, t_o) \\ &= \frac{1}{2}\, \underline{x}_o^T\, [\underline{V}(t_o, \underline{P}) - \underline{K}(t_o)]\underline{x}_o \\ &\triangleq \frac{1}{2}\, \underline{x}_o^T\, \underline{W}(t_o, \underline{P})\underline{x}_o \end{aligned} \tag{34}$$

is a real quadratic form in $\underline{x}_o$, and that the following well-known [16,17] theorem about the extremal properties of the eigenvalues of a real quadratic form is applicable.

Theorem: The global maximum of a real quadratic form on the unit hypersphere is equal to the largest eigenvalue of the quadratic form, and moreover the corresponding eigenvector is the vector drawn from the origin to the point on the hypersphere where the quadratic form achieves its maximum. Therefore

$$\max_{\underline{x}_o \in X} AD(\underline{x}_o, t_o, \underline{P}) = \frac{1}{2}\, \lambda_1(\underline{P}) \tag{35}$$

where $\lambda_1(\underline{P})$ is the largest eigenvalue of $\underline{W}(t_o, \underline{P})$ defined by (34), and the problem is to minimize $\lambda_1(\underline{P})$ with respect

to the elements of $\underline{P}$; that is, to find the constant matrix $\underline{P}_s$ such that

$$J_a(\underline{P}_s) = J_a^*$$

$$= \frac{1}{2} \underset{\underline{P}}{\text{Min}} \; \lambda_1(\underline{P}) \quad . \tag{36}$$

Such a formulation has several advantages over the formulation given by (32). First,

$$\lambda_1(\underline{P}) = 2 \underset{\underline{x}_o \epsilon X}{\text{Max}} \; AD(\underline{x}_o, t_o, \underline{P}) \tag{37}$$

can be evaluated without the need to find the point where the maximum occurs. This is very important because (37) is evaluated many times during the minimization process, and any savings in the number of necessary computations is significant. Second, the programming required is extremely simple because very efficient methods are available for finding the largest eigenvalue of a real symmetric matrix, and many computer facilities have several standard subroutines that can be used for this purpose. However, it should be noted that a subroutine that finds all eigenvalues (and eigenvectors) of a general square matrix is not as suitable for this purpose as one which finds only the largest eigenvalue of a real symmetric matrix.

2. Algorithm for Computing the Constant Output-Feedback Gain Matrix - Algorithm I

a. Calculate $\underline{K}(t_o)$ by integrating (5) backward with the boundary condition (6).

b. Guess starting values, for the entries of $\underline{P}$.

c. Find $\underline{P}$ that minimizes $\lambda_1(\underline{P})$ by using a subroutine that preferably does not require the evaluation of the gradient. Such a subroutine will, in general, require the evaluation of $\lambda_1(\underline{P})$ many times. This can be accomplished as follows:

(1) Calculate $\underline{V}(t_o,\underline{P})$ by integrating (15) backward with the boundary condition (16), using the current value of $\underline{P}$.

(2) Calculate $\underline{W}(t_o,\underline{P})$ using (34).

(3) Find the largest eigenvalue, $\lambda_1(\underline{P})$, of $\underline{W}(t_o,\underline{P})$ using a suitable subroutine.

d. If the location of the point where the worst degradation occurs is desired, then find the eigenvector corresponding to $\lambda_1(\underline{P}_s)$.

In applying this algorithm to the examples given in the rest of this thesis, the minimizations were accomplished using a subroutine[1] that performs the pattern search method of Hooke and Jeeves [18,19]; the largest

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[1] Subroutine DIRECT, NPS Computer Facility.

eigenvalues were found using a subroutine[1] that utilizes Givens-Householder method; and integrations were performed using a Runge-Kutta-Gill fourth-order method[2].

3. Application of Algorithm I - Example 2

The purpose of this example is to illustrate the application of the described algorithm to a simple second-order system, and also to illustrate the importance of using the appropriate algorithm.

The system to be controlled is described by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{38}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} . \tag{39}$$

The performance measure for optimal control is

$$J(\underline{x}_o, u(\cdot)) = \frac{1}{2} \int_o^{\frac{5\pi}{4}} [x_1^2(t) + x_2^2(t) + u^2(t)]dt. \tag{40}$$

The constrained optimal control is to be in the form

$$u_s(t) = - \begin{bmatrix} p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \tag{41}$$

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1 Subroutine GIVHO, see Ralston and Wilf [20].

2 Function RKLDEQ, NPS Computer Facility.

where p_{11} and p_{12} are constants. Expanding the Riccati equation (5) with

$$\underline{A} = \begin{bmatrix} 0 & 1 \\ -3 & -1 \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \underline{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \underline{R} = 1$$

one obtains

$$\begin{aligned} \dot{k}_{11}(t) &= 6k_{12}(t) - 1 + k_{12}^2(t) \\ \dot{k}_{12}(t) &= k_{12}(t) - k_{11}(t) + 3k_{22}(t) + k_{12}(t)k_{22}(t) \\ \dot{k}_{22}(t) &= 2[k_{22}(t) - k_{12}(t)] - 1 + k_{22}^2(t) \end{aligned} \tag{42}$$

with the boundary conditions

$$k_{11}\left(\frac{5\pi}{4}\right) = k_{12}\left(\frac{5\pi}{4}\right) = k_{22}\left(\frac{5\pi}{4}\right) = 0 \ . \tag{43}$$

Note that due to the symmetry of $\underline{K}(t)$ only three equations are needed. Expanding the equation (15) with

$$\underline{P} = [p_{11} \ p_{12}] \quad \text{and} \quad \underline{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

one obtains

$$\begin{aligned} \dot{v}_{11}(t) &= 2(3+p_{11})v_{12}(t) - 1 - p_{11}^2 \\ \dot{v}_{12}(t) &= (1+p_{12})v_{12}(t) - v_{11}(t) + (3+p_{11})v_{22}(t) - p_{11}p_{12} \\ \dot{v}_{22}(t) &= 2(1+p_{12})v_{22}(t) - 2v_{12}(t) - 1 - p_{12}^2 \end{aligned} \tag{44}$$

with boundary conditions

$$v_{11}\left(\frac{5\pi}{4}\right) = v_{12}\left(\frac{5\pi}{4}\right) = v_{22}\left(\frac{5\pi}{4}\right) = 0 \ . \qquad (45)$$

Again due to symmetry only three equations are needed.

The proposed algorithm gives the solution for the constant output-(or state) feedback gain matrix as

$$\underline{P}_s = [0.1537 \quad 0.5097]$$

The normalized degradation (ND) is 0.00055 which shows that the constrained optimal control is almost as good as the optimal control.

Koivuniemi [9] treats the same system as an example for the design of a constrained optimal controller which he calls a specific optimal controller. He uses the same auxiliary performance measure as here, but obtains a result which is quite different. He gives the constant output-(or state) feedback gain matrix as

$$\underline{P}_s = [0.69 \quad 0.76]$$

Assuming that the initial states lie on the unit circle, the optimal performance measure J^*, constrained optimal performance measure J_s, absolute degradation AD, and relative degradation RD are plotted as functions of $x_2(t_o)/x_1(t_o)$ using the $\underline{P}_s$ matrix obtained here and the $\underline{P}_s$ matrix given in [9] in Figs. 6 and 7 respectively, for comparison. Clearly the result found here results in a much better

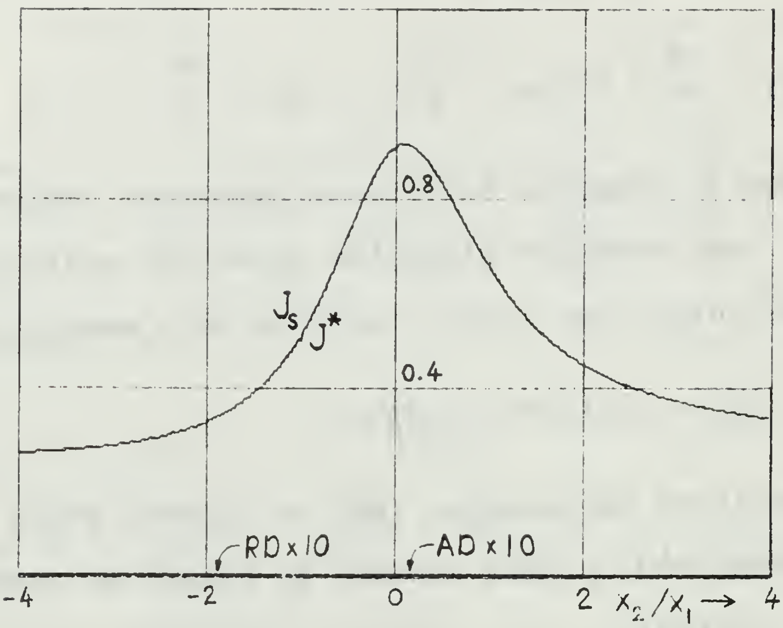

Fig. 6. Various performance values as functions of initial state - using feedback gains found in Example 2.

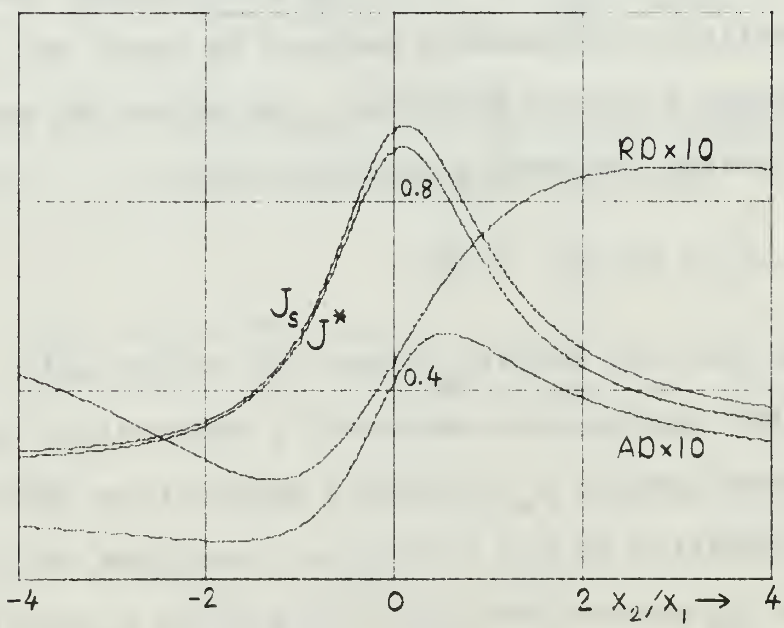

Fig. 7. Various performance values as functions of initial state - using feedback gains given in Ref. 9.

performance for all initial states than the result given by Koivuniemi. This is also obvious from a comparison of the normalized degradation (ND) figures. It is 0.057 for Koivuniemi's design -- much higher than 0.00055 obtained here.

This example has illustrated the importance of using an efficient algorithm for the solution of the minimax problem. Apparently, the algorithm given in [9] has failed at some point, resulting in a solution which is not as good as it could be for this example.

If the output of the same system (38) is given by

$$y_1(t) = [1 \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{46}$$

and the constrained optimal control is specified in the form

$$u_s(t) = - p_{11} y_1(t) \ , \tag{47}$$

then the expansion of (15) with

$$\underline{P} = p_{11} \quad \text{and} \quad \underline{C} = [1 \quad 0]$$

gives

$$\begin{aligned} \dot{v}_{11}(t) &= 2(3+p_{11})v_{12}(t) - 1 - p_{11}^2 \\ \dot{v}_{12}(t) &= v_{12}(t) - v_{11}(t) + (3+p_{11})v_{22}(t) \\ \dot{v}_{22}(t) &= 2v_{22}(t) - 2v_{12}(t) - 1 \end{aligned} \tag{48}$$

with boundary conditions (45). Applying the same algorithm, one obtains

$$\underline{P}_s = p_{11} = -0.236$$

which gives a normalized degradation (ND) of 0.137. Koivuniemi's result for this case is

$$\underline{P}_s = p_{11} = -0.25$$

which agrees reasonably well with the result found here. Evidently, the algorithm given in [9] works in this case.

B. PIECEWISE-CONSTANT OUTPUT-FEEDBACK GAIN MATRIX

In this section, the constrained optimal control problem will be solved for the case where the elements of $\underline{P}(t)$ are constrained to be piecewise-constant in the interval between t_o and t_f .

1. Development of the Method

Suppose that there are N subintervals of time, each of duration Δt, during which the elements of $\underline{P}(t)$ are constant. Let

$$t_i = t_o + i\Delta t, \quad i = 0,1, \ldots , N \tag{49}$$

and let

$$\underline{P}(t) = \underline{P}_i \ , \quad t_i \leq t < t_{i+1} \ , \quad i = 0,1, \ldots , N-1 \tag{50}$$

Since $\underline{P}_i$ is an m x r matrix and since there are N such constant matrices, one can minimize the auxiliary performance measure given by (29) with respect to the mrN elements which are to be determined and obtain the piecewise-constant $\underline{P}_s(t)$. However if mrN is a large number, the minimization of the auxiliary performance measure with respect to such a large number of variables may be a very difficult and time-consuming task.

A much simpler, yet possibly more powerful and practical approach will be used here. This approach is inspired by the Principle of Optimality of Bellman [1]. Suppose that the time t_{N-1} has been reached and there remains only one interval during which control can be applied. At this point, if $\underline{x}(t_{N-1})$ were known a priori, then to determine $\underline{P}_{N-1}$ would be a simple parameter optimization problem, and $\underline{P}_{N-1}$ would be determined to minimize $J(\underline{x}(t_{N-1}), t_{N-1}, \underline{P}_{N-1})$. However, $\underline{x}(t_{N-1})$ is not known a priori, therefore, one is again confronted with the problem of defining an auxiliary performance measure. Fortunately, the solution proposed for the constant-$\underline{P}$ case in the previous section can be applied to this case also by simply taking t_{N-1} as the initial time and applying the Algorithm I. This gives the elements of $\underline{P}_{N-1}$ as well as $\underline{K}(t_{N-1})$, $\underline{V}(t_{N-1})$ and $\underline{W}(t_{N-1})$. This solution for the last interval is equivalent to assuming that the admissible $\underline{x}(t_{N-1})$ lies on the unit hypersphere (without loss of generality) and minimizing the maximum absolute degradation with respect to the admissible

$\underline{x}(t_{N-1})$. Similarly, $\underline{P}_{N-2}$ can be determined using t_{N-2} and t_{N-1} as the initial and the final times respectively and using $\underline{K}(t_{N-1})$ and $\underline{V}(t_{N-1})$ obtained as a result of the previous application as the boundary conditions in performing the integrations. The same procedure is applied to all subintervals until $\underline{P}_o$, $\underline{K}(t_o)$ and $\underline{V}(t_o)$ are obtained at the last application.

2. <u>Algorithm for Computing the Piecewise-Constant Output-Feedback Gain Matrix - Algorithm II</u>

a. Let $i = N$.

b. Let $t_f = t_i$ and $t_o = t_{i-1}$.

c. Apply Algorithm I to obtain $\underline{P}_{i-1}$, $\underline{K}(t_{i-1})$ and $\underline{V}(t_{i-1})$.

d. If $i = 1$, go to f.

e. Let $i = i - 1$, go to b.

f. Stop.

3. <u>Some Comments Concerning the Application of Algorithm II</u>

The optimal value for the auxiliary performance measure, J_a^* , obtained by applying Algorithm II will be greater than the value that would be obtained if the minimization were accomplished at one step with respect to mrN elements as indicated earlier. However, it should be remembered that in the latter case the "worst" initial state (the initial state that causes the maximum absolute degradation) is assumed at time t_o only and not at the beginning of

each subinterval, whereas, in the method of solution proposed here, the constant output-feedback gain matrix for each subinterval is determined to minimize the maximum absolute degradation at the beginning of the subinterval.

It is even possible that the normalized degradation may increase with a small increase in the number of subintervals. This is because the "worst" is assumed to occur more often and the parameter values that are chosen with this in mind may result in higher normalized degradation than the case where "worst" is assumed less often or just at the beginning. However, when this occurs, it still does not mean that increasing the number of subintervals is not advantageous. This entirely depends on the application. If the system to be controlled is in an environment which is known very well and no disturbances are anticipated, then increasing the number of subintervals in the manner proposed here does not result in a "better" system, unless there is a decrease in normalized degradation. If piecewise-constant output-feedback gains are desired in such a case, then one should perform the minimization at one step with respect to all mrN elements as mentioned earlier. However, in practice many systems operate in environments where unpredictable disturbances can occur at any time. In those cases, increasing the number of subintervals even when the normalized degradation does not decrease is desirable, at least from a conservative design point of view.

When all of the states of a system are available as outputs and the number of subintervals approaches infinity, i.e., the constraints are relaxed, then one expects the constrained optimal control to approach the optimal control and the normalized degradation to approach zero. This was the case in all the examples studied by the author.

4. Application of Algorithm II

a. Example 3

The purpose of this example is to illustrate the application of Algorithm II to a third-order system. The system[1] to be controlled is described by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} u(t) \tag{51}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} . \tag{52}$$

The performance measure for optimal control is

$$J(\underline{x}_o, u(\cdot)) = \int_o^2 \left\{ [x_1(t) x_2(t) x_3(t)] \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + u^2(t) \right\} dt. \tag{53}$$

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[1] This system is treated in Ref. 12 where constant and piecewise-constant gains are computed using a different auxiliary performance measure.

The constrained optimal control is to be in the form

$$u_s(t) = -[p_{11}(t)\ p_{12}(t)\ p_{13}(t)] \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}, \qquad (54)$$

where $[p_{11}(t)\ p_{12}(t)\ p_{13}(t)]$ is a piecewise-constant output-feedback gain matrix. The matrices needed for the application of Algorithms I and II are:

$$\underline{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix} \qquad \underline{B} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \qquad \underline{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{Q} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \underline{R} = 2 \qquad \underline{P} = [p_{11}(t)\ p_{12}(t)\ p_{13}(t)]$$

and $\underline{H} = 0$. This problem was solved for the one-subinterval (a constant feedback gain matrix) case using Algorithm I, and for two and ten-subinterval (piecewise-constant feedback gain matrices) cases using Algorithm II. The elements of $\underline{F}_s(t) = \underline{P}_s(t)\underline{C}$ and $\underline{F}^*(t) = \underline{R}^{-1}\underline{B}^T\underline{K}(t)$ are plotted in Figs. 8, 9, and 10 for each case. The normalized degradation is 0.0356 for the one-subinterval case and decreases to the values 0.0135 and 0.0025 for the two and ten-subinterval cases, respectively. As the number of subintervals is further increased, the constrained optimal feedback gains become closer to the optimal feedback gains, and the

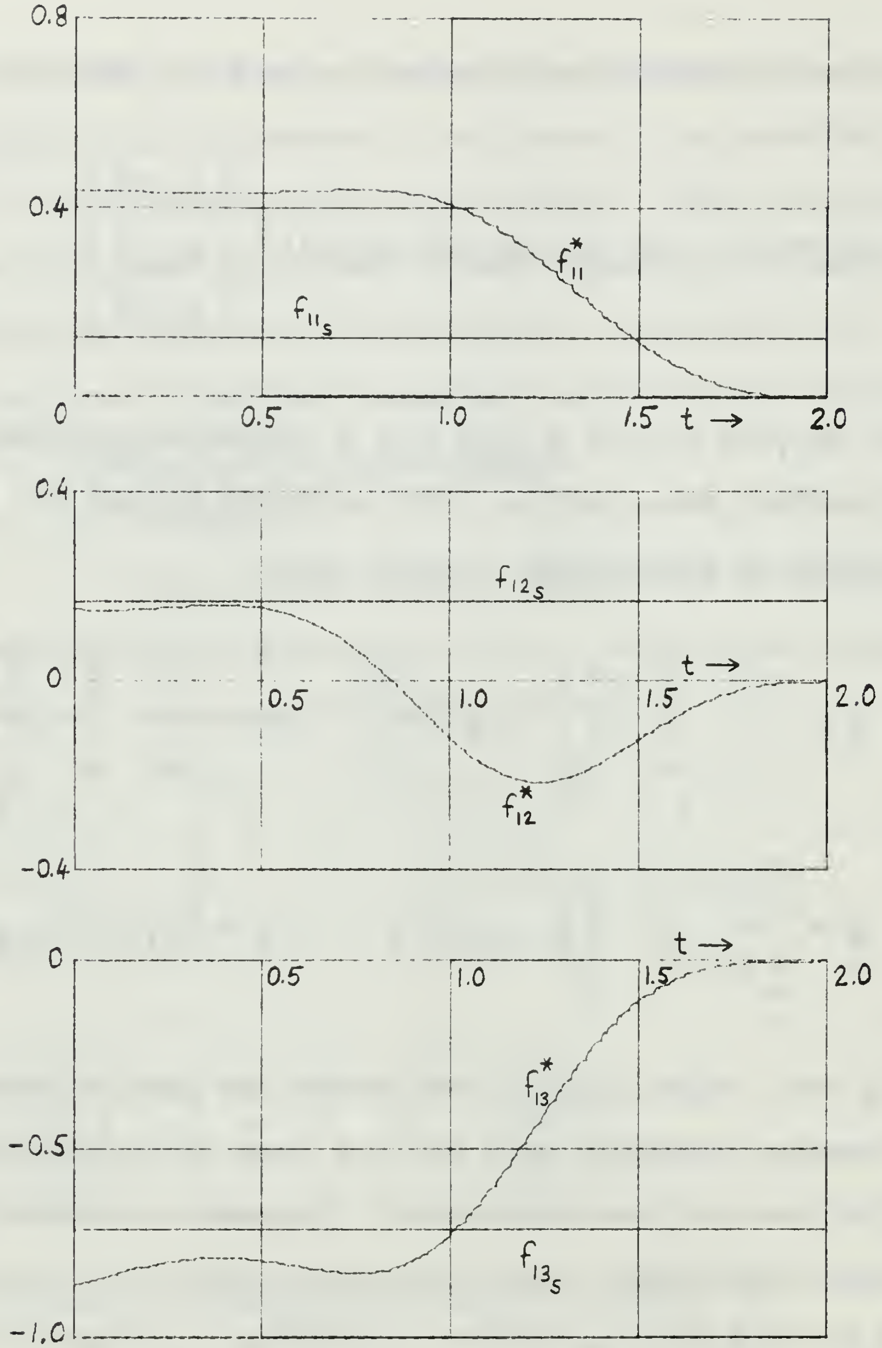

Fig. 8. Optimal and constrained optimal feedback gains for Example 3 -- One subinterval.

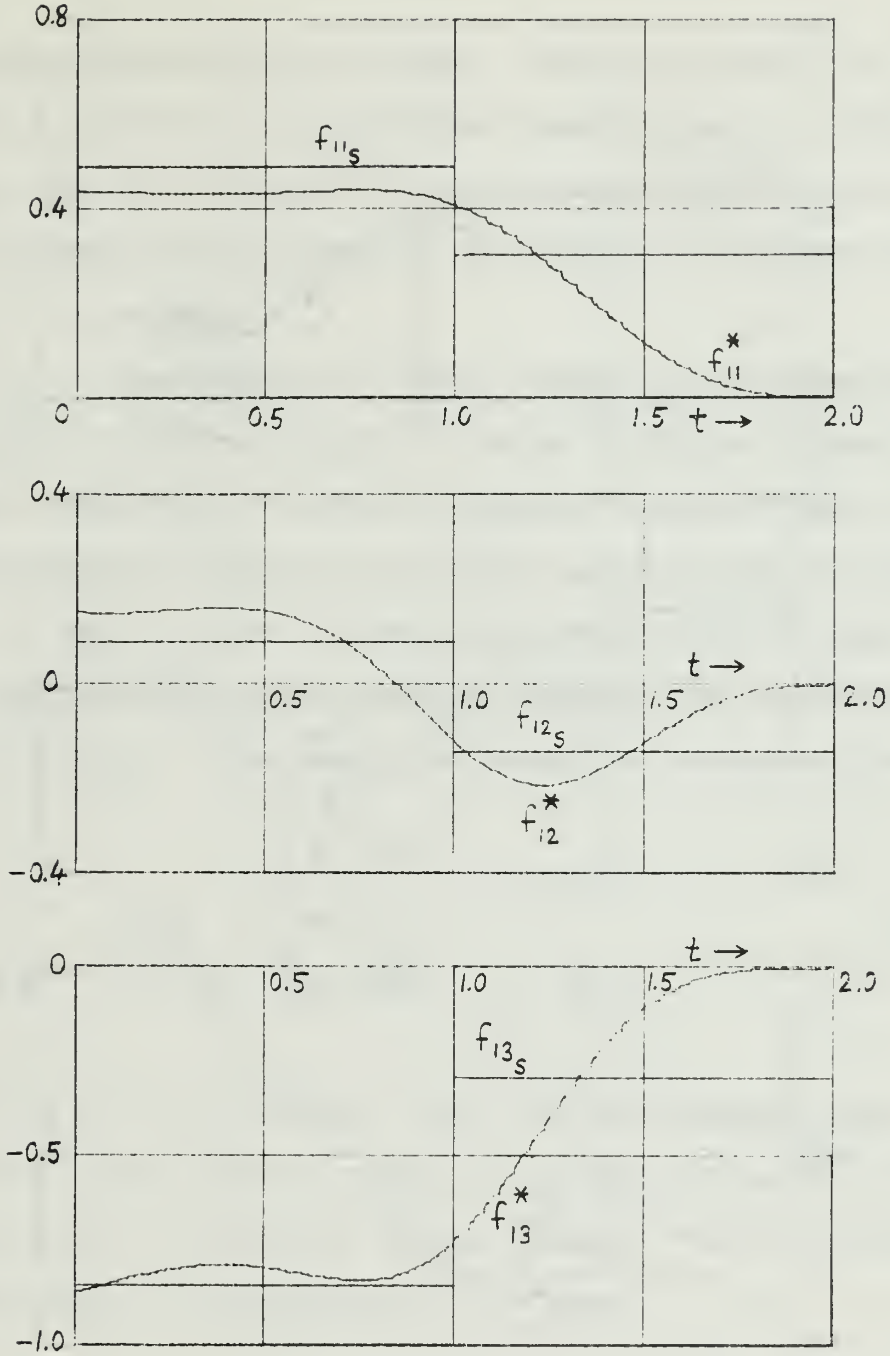

Fig. 9. Optimal and constrained optimal feedback gains for Example 3 -- Two subintervals.

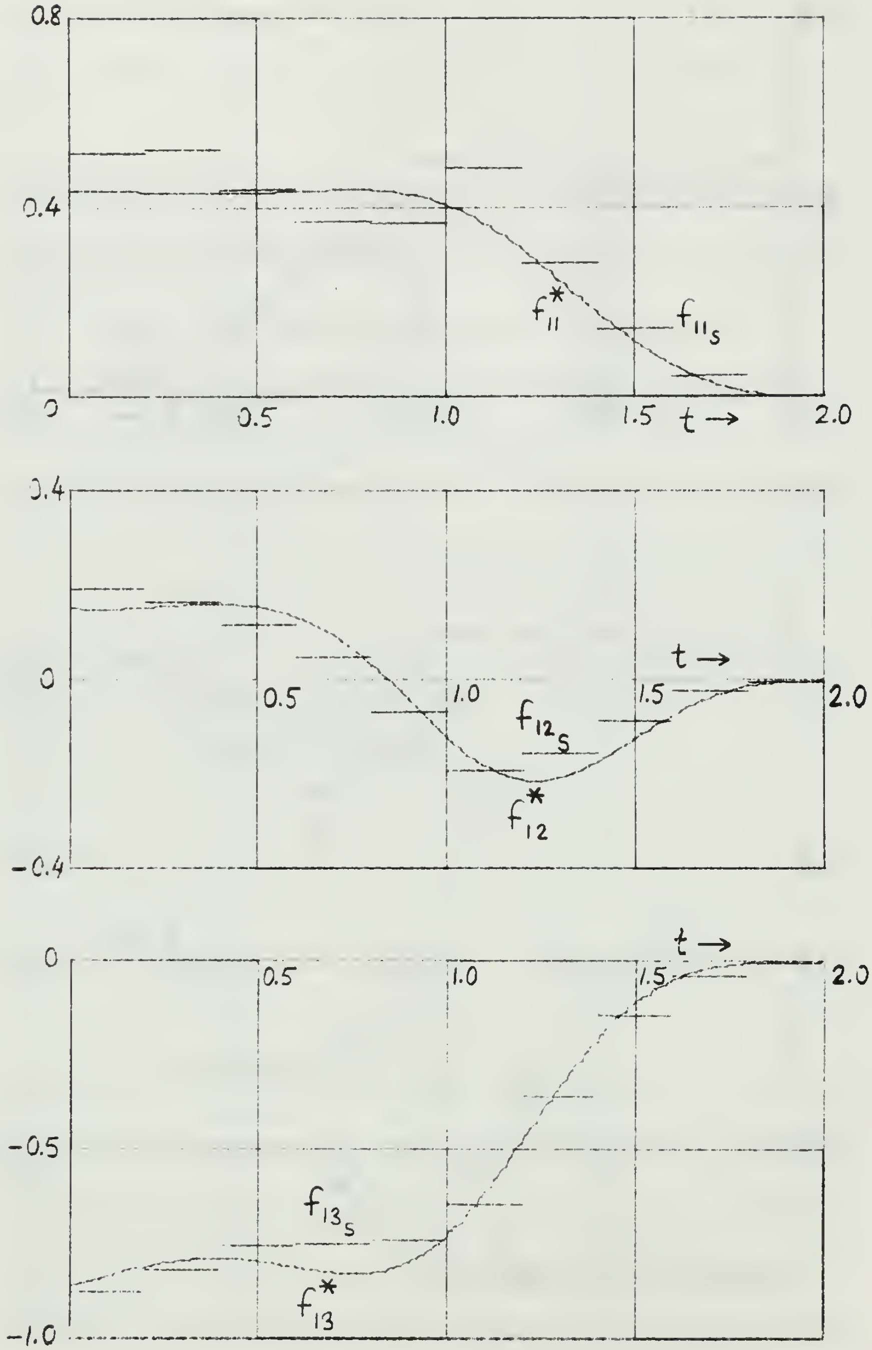

Fig. 10. Optimal and constrained optimal feedback gains for Example 3 -- Ten subintervals.

normalized degradation decreases. When the number of subintervals is fifty the normalized degradation is 0.0002. Clearly, for this example, one can get as close to the optimal as desired by increasing the number of subintervals.

b. Example 4

The purpose of this example is to investigate and compare the effectiveness of using different states of a linear regulator to form the feedback control, and to provide another illustration of the applications of Algorithms I and II. The system to be controlled is again given by (51) and the performance measure for the optimal control by (53). This time the output is a scalar given by

$$y(t) = \underline{C} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \tag{55}$$

where $\underline{C}$ is a 1×3 matrix, with only one element equal to one and the rest equal to zero. In other words, only one of the states is available as an output. The constrained optimal control is specified in the form

$$u_s(t) = - p(t)y(t) \tag{56}$$

where $p(t)$ is the piecewise-constant output-feedback gain. Table 2 gives the normalized degradations for different outputs and for various numbers of subintervals. Clearly, $C = [0 \quad 0 \quad 1]$ results in the best constrained optimal

control, i.e., $x_3(t)$ is quite effective in forming the control. Considering that the normalized degradation is 0.5653 without any control ($u(t) \equiv 0$) it can be seen that $C = [1\ 0\ 0]$ or $C = [0\ 1\ 0]$ provide very little improvement over no control. When $C = [0\ 0\ 1]$, there is a small increase in the normalized degradation when the number of subintervals is increased from one to two. This was mentioned as a possibility previously. However, further increase in the number of subintervals decreases the normalized degradation again.

$\underline{C}$	One Subinterval	Two Subintervals	Four Subintervals
[1 0 0]	0.5022	0.5022	0.5022
[0 1 0]	0.5252	0.4767	0.4419
[0 0 1]	0.0907	0.0942	0.0822

Table 2. Normalized degradations for different outputs and number of subintervals for Example 4.

c. Example 5

In this example, Algorithms I and II are applied to a second-order, time-varying system. The system is described by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{25-t} & \frac{1}{25-t} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.08167 \end{bmatrix} u(t) \qquad (57)[1]$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} . \qquad (58)$$

The performance measure for optimal control is

$$J(\underline{x}_o, u(\cdot)) = \frac{1}{2} \int_o^{10} [200\, x_1^{\,2}(t) + 2u^2(t)]dt. \qquad (59)$$

The constrained optimal control is to be in the form

$$u_s(t) = -\,[p_{11}(t)\ \ p_{12}(t)] \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \qquad (60)$$

where $[p_{11}(t)\ p_{12}(t)]$ is a piecewise-constant output-feedback gain matrix. Expanding the equations (5) and (15) with

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[1] These equations represent the perturbed motion of the flight of an airplane in a vertical plane on a gliding path inside the equisignal zone of the glide radio beacon, with constant velocity. For more detail on the description of the system, see Example 1 of Ref. 21 where an optimal controller is designed for the system.

$$\underline{A} = \begin{bmatrix} \frac{1}{25-t} & \frac{1}{25-t} \\ 0 & 0 \end{bmatrix} \qquad \underline{B} = \begin{bmatrix} 0 \\ 0.08167 \end{bmatrix}$$

$$\underline{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \underline{Q} = \begin{bmatrix} 200 & 0 \\ 0 & 0 \end{bmatrix}$$

$\underline{R} = 2$ and $\underline{H} = 0$ and applying Algorithms I and II, the problem was solved for several cases with various numbers of subintervals. The elements of $\underline{F}_s(t) = \underline{P}_s(t)\underline{C}$ and $\underline{F}^*(t) = \underline{R}^{-1}\underline{B}^T\underline{K}(t)$ are plotted in Figs. 11 and 12 for one and two-subinterval cases respectively. The normalized degradation is 0.02 for the one-subinterval case and 0.0852 for the two-subinterval case. This increase in the normalized degradation with an increase in the number of subintervals was mentioned previously as a possibility in the comments following the description of Algorithm II. However, as the number of subintervals is increased further, the normalized degradation decreases and approaches zero, e.g., it is 0.0205 for the ten-subinterval case and 0.0015 for the fifty-subinterval case. This example shows that the application of Algorithms I and II to time-varying systems is as straightforward as the application to time-invariant systems.

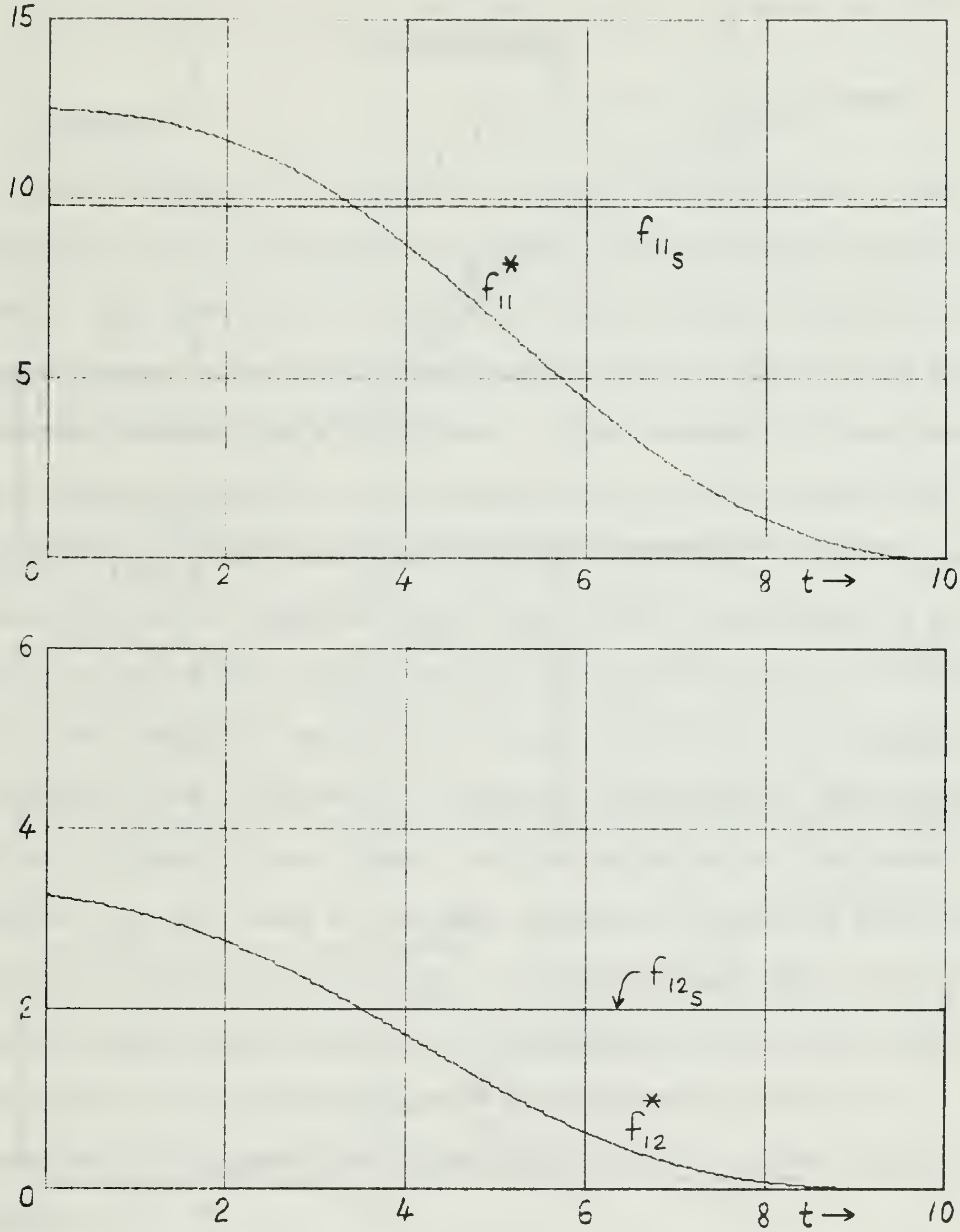

Fig. 11. Optimal and constrained optimal feedback gains for Example 5 -- One subinterval.

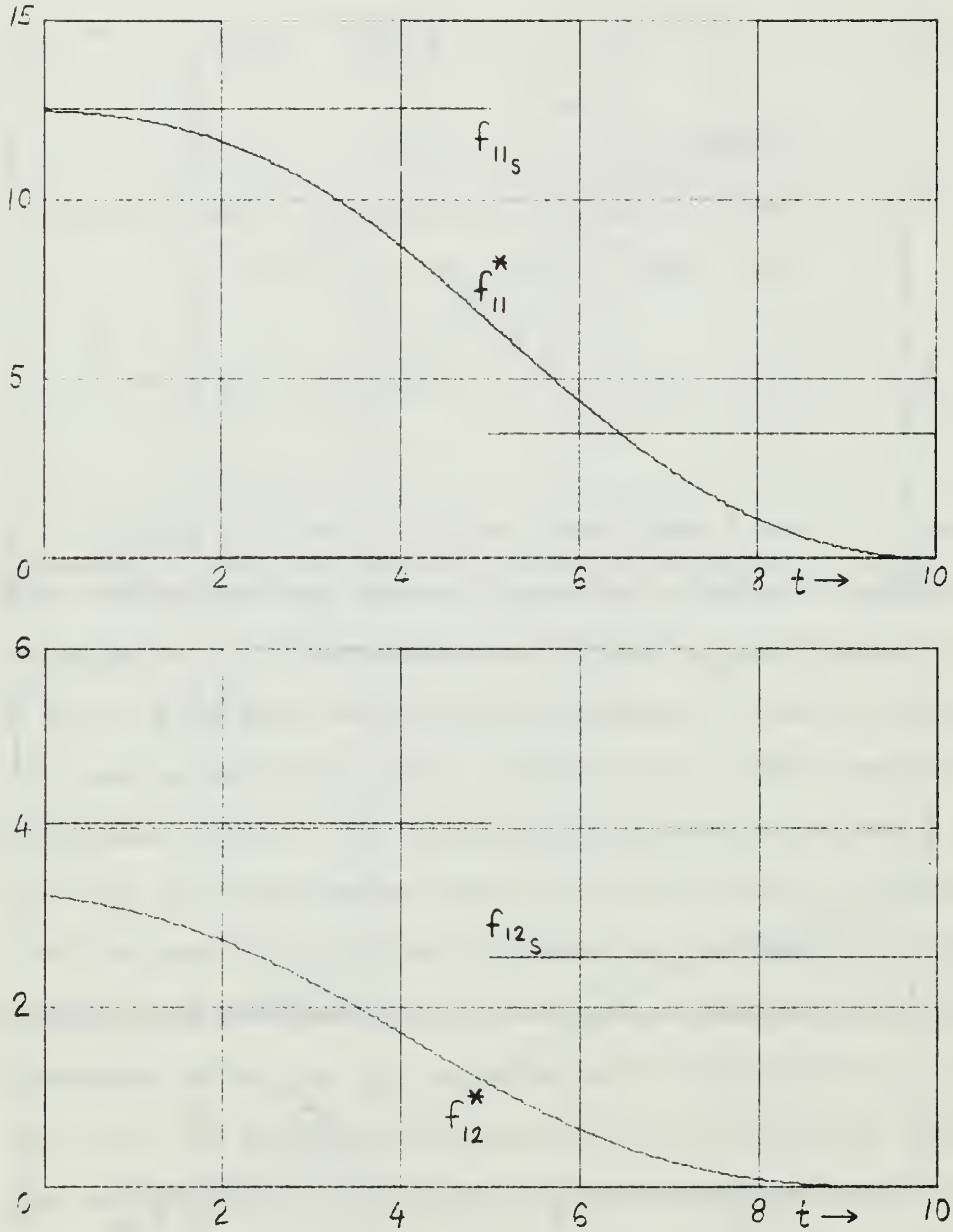

Fig. 12. Optimal and constrained optimal feedback gains for Example 5 -- Two subintervals.

V. CONCLUSIONS

A. SUMMARY

The problem of designing optimal controls for linear regulators under complexity constraints has been investigated. The need for an auxiliary performance measure was demonstrated and several candidates for the auxiliary performance measure were compared. The maximum of the absolute degradation over the admissible initial states was selected as the auxiliary performance measure. A method was developed and a computational algorithm (Algorithm I) was given to solve the resulting minimax problem for the case where the feedback gains are constrained to be constant. A conservative engineering approach inspired by the Principle of Optimality was used for the solution of the same problem for the case where the feedback gains are constrained to be piecewise-constant. It was assumed that the "worst" initial state could occur at the beginning of each subinterval and the parameters were determined accordingly. A computational algorithm (Algorithm II) was given for this purpose.

Both algorithms were programmed in FORTRAN IV for the IBM 360 in use at the NPS Computer Facility. Since the algorithms rely heavily upon existing, well-established library subprograms, the programming required was very simple, yet efficient and reliable. In none of the problems

solved did the results depend on the initial guess for the parameter values. For the minimization step of the algorithms a gradient scheme in which partial derivatives were evaluated by making small perturbations was also tried and gave the same results as the pattern search in all cases. The gradient algorithm, however, required using more than one initial guess in some instances, and did not improve the computation time observed with the pattern search method.

Algorithm II does not add much complexity to Algorithm I, nor does it increase the computation time. This is because most of the computation time is used for integration, and when the number of subintervals is increased, there is a corresponding decrease in the number of integration steps per subinterval and hence no significant change in computation time occurs.

B. PROBLEMS FOR FURTHER STUDY

It would be worthwhile to investigate the application of the given algorithms to linear tracking problems because these problems have wide applications and methods for easily realizable controllers for such systems are very desirable.

Another worthwhile extension is the application of the given methods to the design of simple compensators for automatic control systems. For example, for a third-order linear regulator with only one output, a second-order

compensator could be combined with the system and then in addition to the output feedback gain, the compensator parameters could be optimized in a minimax sense for the overall fifth-order system.

APPENDIX

DERIVATION[1] OF EQUATIONS (14),(15) AND (16)

For the system and the performance measure given by (1), (2) and (3), consider the control law given by (10) or (11). Substituting (11) in (1), one obtains

$$\dot{\underline{x}}(t) = [\underline{A}(t) - \underline{B}(t)\underline{F}(t)]\underline{x}(t). \qquad (A.1)$$

Let $\underline{\Phi}_s(t,t_o)$ be the transition matrix [1] of (A.1), then the motion of (A.1) is given by

$$\underline{x}(t) = \underline{\Phi}_s(t,t_o)\underline{x}_o \qquad (A.2)$$

and $\underline{\Phi}_s(t,t_o)$ satisfies the differential equation

$$\frac{d}{dt}\,\underline{\Phi}_s(t,t_o) = [\underline{A}(t) - \underline{B}(t)\underline{F}(t)]\underline{\Phi}_s(t,t_o) \qquad (A.3)$$

with the boundary condition $\underline{\Phi}_s(t_o,t_o) = \underline{I}$. In addition, the final state is related to any preceding state by

$$\underline{x}(t_f) = \underline{\Phi}_s(t_f,t)\underline{x}(t)\ , \qquad (A.4)$$

and, in particular, to the initial state by

$$\underline{x}(t_f)\ \underline{\Phi}_s(t_f,t_o)\underline{x}_o\ . \qquad (A.5)$$

$\underline{\Phi}_s(t_f,t)$ satisfies[2] the differential equation

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[1] This derivation follows the outline given in [8].

[2] This can be shown by differentiating (A.4) with respect to t and noting that $d\underline{x}(t_f)/dt = 0$.

$$\frac{d}{dt}\,\underline{\Phi}_s(t_f,t) = -\underline{\Phi}_s(t_f,t)[\underline{A}(t)-\underline{B}(t)\underline{F}(t)] \tag{A.6}$$

for all t_f and t, with the boundary condition $\underline{\Phi}_s(t_f,t_f)=\underline{I}$. Substituting (11) in (3), one obtains

$$J(\underline{x}_o,t_o,\underline{P}(\cdot)) = \frac{1}{2}\,\underline{x}^T(t_f)\underline{H}\,\underline{x}(t_f) + \frac{1}{2}\int_{t_o}^{t_f} \underline{x}^T(t)[\underline{Q}(t)+\underline{F}^T(t)\underline{R}(t)\underline{F}(t)]\underline{x}(t)\,dt. \tag{A.7}$$

Further substitution for $\underline{x}(t)$ and $\underline{x}(t_f)$ using (A.2) and (A.5) gives

$$J(\underline{x}_o,t_o,\underline{P}(\cdot)) = \frac{1}{2}\,\underline{x}_o{}^T\underline{\Phi}_s{}^T(t_f,t_o)\underline{H}\,\underline{\Phi}_s(t_f,t_o)\underline{x}_o + \frac{1}{2}\int_{t_o}^{t_f} \underline{x}_o{}^T\,\underline{\Phi}_s^T(t,t_o)[\underline{Q}(t)+\underline{F}^T(t)\underline{R}(t)\underline{F}(t)]\underline{\Phi}_s(t,t_o)\underline{x}_o\,dt. \tag{A.8}$$

Combining the two terms gives

$$J(\underline{x}_o,t_o\underline{P}(\cdot)) = \frac{1}{2}\,\underline{x}_o^T\left\{\underline{\Phi}_s{}^T(t_f,t_o)\underline{H}\,\underline{\Phi}_s(t_f,t_o) + \int_{t_o}^{t_f} \underline{\Phi}_s^T(t,t_o)[\underline{Q}(t)+\underline{F}^T(t)\underline{R}(t)\underline{F}(t)]\underline{\Phi}_s(t,t_o)\,dt\right\}\underline{x}_o\;. \tag{A.9}$$

Defining

$$\underline{V}(t) \triangleq \underline{\Phi}_s{}^T(t_f,t)\underline{H}\,\underline{\Phi}_s(t_f,t) + \int_{t}^{t_f} \underline{\Phi}_s^T(\tau,t)[\underline{Q}(\tau)+\underline{F}^T(\tau)\underline{R}(\tau)\underline{F}(\tau)]\underline{\Phi}_s(\tau,t)\,d\tau, \tag{A.10}$$

then $\underline{V}(t_o,\underline{P}(\cdot))$[1] is the term in the braces in (A.9) and

$$J(\underline{x}_o,t_o,\underline{P}(\cdot)) = \frac{1}{2}\,\underline{x}_o^T\underline{V}(t_o,\underline{P}(\cdot))\underline{x}_o\ , \tag{14}$$

which is repeated here for convenience, is obtained.

To show that (15) and (16) hold, differentiate[2] (A.10) with respect to t, with the result

$$\begin{aligned}\dot{\underline{V}}(t) &= \dot{\underline{\Phi}}_s^T(t_f,t)\underline{H}\ \underline{\Phi}_s(t_f,t) + \underline{\Phi}_s^T(t_f,t)\underline{H}\ \dot{\underline{\Phi}}(t_f,t)\\ &+ \int_t^{t_f}\Big\{\dot{\underline{\Phi}}_s^T(\tau,t)[\underline{Q}(\tau)+\underline{F}^T(\tau)\underline{R}(\tau)\underline{F}(\tau)]\underline{\Phi}_s(\tau,t)\\ &\quad + \underline{\Phi}_s^T(\tau,t)[\underline{Q}(\tau)+\underline{F}^T(\tau)\underline{R}(\tau)\underline{F}(\tau)]\dot{\underline{\Phi}}_s(\tau,t)\Big\}\,d\tau\\ &\quad - [\underline{Q}(t)+\underline{F}^T(t)\underline{R}(t)\underline{F}(t)]\ .\end{aligned} \tag{A.11}$$

Substituting (A.6) for $\dot{\underline{\Phi}}_s(t_f,t)$ and $\dot{\underline{\Phi}}_s(\tau,t)$ (noting the necessary correspondence between t_f and τ in the latter case) gives

---

[1] See the footnote on page 17.

[2] If $I(t) = \int_t^{t_f} f(\tau,t)d\tau$, then by Leibnitz's rule

$$\frac{dI(t)}{dt} = \int_t^{t_f}\frac{\partial f(\tau,t)}{\partial t}\,d\tau - f(t,t).$$

$$\underline{\dot{V}}(t) = -[\underline{A}(t) - \underline{B}(t)\underline{F}(t)]^T \underline{\Phi}_s^T(t_f, t)\underline{H}\ \underline{\Phi}_s(t_f, t)$$

$$- \underline{\Phi}_s{}^T(t_f, t)\underline{H}\ \underline{\Phi}_s(t_f, t)[\underline{A}(t) - \underline{B}(t)\underline{F}(t)]$$

$$- \int_t^{t_f} [\underline{A}(t) - \underline{B}(t)\underline{F}(t)]^T \underline{\Phi}_s^T(\tau, t)[\underline{Q}(\tau) + \underline{F}^T(\tau)\underline{R}(\tau)\underline{F}(\tau)]\underline{\Phi}_s(\tau, t)d\tau$$

$$- \int_t^{t_f} \underline{\Phi}_s^T(\tau, t)[\underline{Q}(\tau) + \underline{F}^T(\tau)\underline{R}(\tau)F(\tau)]\underline{\Phi}_s(\tau, t)[\underline{A}(t) - \underline{B}(t)\underline{F}(t)]d\tau$$

$$- \underline{Q}(t) - \underline{F}^T(t)\underline{R}(t)\underline{F}(t)\ . \qquad (A.12)$$

Combining the first and third terms and similarly the second and fourth terms of (A.12), and identifying $\underline{V}(t)$ as given by (A.10) one obtains

$$\underline{\dot{V}}(t) = -[\underline{A}(t) - \underline{B}(t)\underline{F}(t)]^T\ \underline{V}(t)$$

$$- \underline{V}(t)[\underline{A}(t) - \underline{B}(t)\underline{F}(t)]$$

$$- \underline{Q}(t) - \underline{F}^T(t)\underline{R}(t)\underline{F}(t) \qquad (A.13)$$

(15) is obtained by substituting $\underline{F}(t) = \underline{P}(t)\underline{C}(t)$ in (A.13); and (16) is obtained by letting $t = t_f$ in (A.10). Taking the transpose of both sides of (A.10) does not change the right side, hence $\underline{V}(t)$ is symmetric. That $\underline{V}(t_o)$ is positive definite can be seen by substituting (14) and (8) in (13) and noting that $\underline{K}(t_o)$ is positive definite.

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UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Naval Postgraduate School Monterey, California 93940	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Design of Constrained Optimal Controls for Linear Regulators with Incomplete State Feedback

4. DESCRIPTIVE NOTES *(Type of report and, inclusive dates)*

Degree of Electrical Engineer Thesis; October 1969

5. AUTHOR(S) *(First name, middle initial, last name)*

Ergin Özer

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
October 1969	66	21

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d.	

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11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Naval Postgraduate School Monterey, California 93940

13. ABSTRACT

The need for an auxiliary performance measure for the design of constrained optimal controls for linear regulators is shown. Several auxiliary performance measures are compared, and the maximum of the absolute degradation over the admissible initial states is selected as the auxiliary performance measure. Computational algorithms which make extensive use of existing library subprograms are developed for the design of constrained optimal controls in those cases where the control vector is specified as constant or piecewise-constant linear feedback of the output vector. Numerical examples including a third-order system and a time-varying system are given to illustrate the applications of the proposed algorithms.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6811

Unclassified

Security Classification

A-31408

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Linear Regulator						
Optimal Control						
Constrained Optimal Control						
Incomplete State Feedback						
Minimax Controller						
Auxiliary Performance Measure						

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S/N 0101-807-6821